Noisy Quantum Metrology Enhanced by Continuous Nondemolition Measurement

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We show that continuous quantum nondemolition (QND) measurement of an atomic ensemble is able to improve the precision of frequency estimation even in the presence of independent dephasing acting on each atom. We numerically simulate the dynamics of an ensemble with up to $N = 150$ atoms initially prepared in a (classical) spin coherent state, and we show that, thanks to the spin squeezing dynamically generated by the measurement, the information obtainable from the continuous photocurrent scales superclassically with respect to the number of atoms N. We provide evidence that such superclassical scaling holds for different values of dephasing and monitoring efficiency. We moreover calculate the extra information obtainable via a final strong measurement on the conditional states generated during the dynamics and show that the corresponding ultimate limit is nearly achieved via a projective measurement of the spin-squeezed collective spin operator. We also briefly discuss the difference between our protocol and standard estimation schemes, where the state preparation time is neglected.

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Quantum enhanced metrology [1,2] is one of the most promising and well developed ideas in the realm of quantum technologies, with application ranging from the probing of delicate biological systems [3] to the squeezing enhanced optical interferometry [4,5] recently exploited in gravitational wave detectors [6,7]. Atom-based quantum enhanced sensors [8,9] have also been intensively studied and have myriad of potential applications [10], most notably in magnetometry [11–15] and atomic clocks [16–18].

Continuous measurements [19,20] have proven to be very useful tools for the exquisite control of quantum systems, a necessary requirement for the realization of quantum technologies. The genuinely quantum regime of observing single trajectories has been reached in different platforms, such as superconducting circuits [21–23], optomechanical [24,25], and hybrid [26] systems. Crucially, continuously monitoring a quantum system allows for the estimation of its characteristic parameters. Literature has emerged discussing both practical estimation strategies [27–37] and the fundamental statistical tools to assess the achievable precision [38–45].

Being also particularly robust against noise [46], spin squeezing [47,48] of atomic ensembles has been long studied as a resource for quantum enhanced metrology. Implementing a continuous quantum nondemolition (QND) measurement of a collective spin observable is a well-known approach to generate a conditional spin-squeezed state and the prototypical realization of such schemes relies on the collective interaction between light and atoms [49–52]. Several measurement-based schemes have been experimentally realized on large atomic ensembles, witnessing spin squeezing of up to $N \approx 10^{11}$ atoms [53–60].

In the ideal noiseless scenario, continuous QND measurements allow one to overcome projection noise and to achieve estimation with Heisenberg limited uncertainty, i.e., inversely proportional to the number of atoms, just by processing the continuous detected signal [29,45,61–64]. In conventional metrological schemes exploiting an initial entangled state, Heisenberg scaling is lost in the presence of most kinds of independent noises [65–68]. If the external degrees of freedom causing the noise can be continuously observed, however, its effect can be (at least partially) counteracted and the usefulness of the initial entangled state preserved [69–72]. The effect of independent noises on continuous QND strategies, in which the entanglement is created dynamically, has not been explored and it will be the main focus of this work. In more detail, we have the following goals: (i) verify if an enhancement is still observed comparing to the situation where no continuous monitoring is performed; (ii) verify if a quantum enhancement due to nonclassical correlations such as spin squeezing and entanglement can still be observed.

Quantum metrology via continuous QND monitoring in the presence of dephasing.—We consider the following scenario: an ensemble of N two-level atoms (qubits) is rotating around the z axis with angular frequency ω; each atom is subjected to equal and independent Markovian dephasing with rate κ, leading to the following Lindblad master equation

$$\frac{d\varrho}{dt} = \mathcal{L}\varrho \equiv -i\omega[J_z, \varrho] + \frac{\kappa}{2}\sum_{j=1}^{N}\mathcal{D}[\sigma_z^{(j)}]\varrho, \quad (1)$$

where $J_z = \sum_{j=1}^{N}\sigma_z^{(j)}/2$, $\mathcal{D}[A]\varrho = A\varrho A^\dagger - (A^\dagger A\varrho + \varrho A^\dagger A)/2$. Our aim is the estimation of the frequency ω, which in optical magnetometry corresponds to the Larmor frequency $\omega = \gamma B$ (γ being the gyromagnetic ratio), thus equivalent to the estimation of the intensity B of a magnetic field.

For noisy quantum frequency estimation schemes, the ultimate limit on the estimation uncertainty $\delta\omega$ for an experiment of total duration T, optimized over the duration t of a single experiment repeated $M = T/t$ times, is given by a quantum Cramér-Rao bound (CRB) of the form [65]

$$(\delta\omega^2)T \geq \frac{1}{\max_t[\mathcal{Q}/t]}, \quad (2)$$

where $\mathcal{Q}$ corresponds to the quantum Fisher information (QFI) of the quantum state evolved up to time t (see Supplemental Material [73] for more details on estimation theory [74–78]).

If the initial state is prepared in a coherent spin state (CSS), i.e., the tensor products of eigenstates of the single atom Pauli matrices $\sigma_x^{(j)}$, $|\psi_{\mathsf{CSS}}\rangle = \otimes_{j=1}^{N}(|0\rangle_j + |1\rangle_j)/\sqrt{2}$, the state remains separable at all times. The QFI of the CSS state, optimized over the monitoring time t, follows the *standard quantum limit* (SQL), i.e., it is linear in N (corresponding to $\delta\omega \sim 1/\sqrt{N}$ for the uncertainty) and reads

$$\mathcal{Q}^\star_{\mathsf{CSS}} \equiv \max_t[\mathcal{Q}_{\mathsf{CSS}}/t] = \frac{N}{2e\kappa}. \quad (3)$$

By allowing initial entangled states, such as a GHZ state $|\psi_{\mathsf{GHZ}}\rangle = (\otimes_{j=1}^{N}|0\rangle_j + \otimes_{j=1}^{N}|1\rangle_j)/\sqrt{2}$, one can achieve a Heisenberg scaling of the QFI, i.e., $\mathcal{Q} \sim N^2$ in the noiseless scenario ($\kappa = 0$).

This quantum enhancement is, however, lost as soon as some nonzero dephasing acts on the system [65–68]. Dephasing is the most detrimental among independent noise channels: remarkably enough, the change of scaling is observed not only asymptotically, but also at finite N [65], and most of the approaches suggested in the literature to circumvent the no-go theorems for noisy quantum metrology are useful only in the presence of noise transverse to the Hamiltonian [98–102] or for time-correlated dephasing [103,104].

We now assume to prepare the atoms in a CSS state $|\psi_{\mathsf{CSS}}\rangle$ at time $t = 0$, and to perform a continuous monitoring of the collective spin operator $J_y = \sum_{j=1}^{N}\sigma_y^{(j)}/2$,

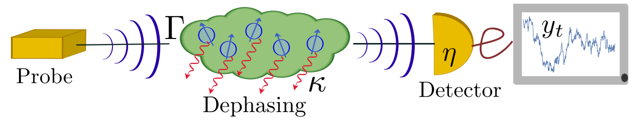

FIG. 1. Quantum magnetometry via continuous measurements: An atomic ensemble of N atoms is sensing a magnetic field that causes precession of the spin around the z axis, and is subjected to independent dephasing on each atom with strength κ. A far-detuned laser shines the atoms, collectively coupling to the total spin J_y with a strength Γ, and it is then measured continuously with efficiency η.

such that the conditional dynamics of the atom ensemble is described by the stochastic master equation (SME)

$$d\varrho_c = \mathcal{L}\varrho_c dt + \Gamma\mathcal{D}[J_y]\varrho_c dt + \sqrt{\eta\Gamma}\mathcal{H}[J_y]\varrho_c dw, \quad (4)$$

conditioned by the measured photocurrent

$$dy_t = 2\sqrt{\eta\Gamma}\mathrm{Tr}[\varrho_c J_y]dt + dw. \quad (5)$$

The parameter Γ corresponds to the collective measurement strength, η to the measurement efficiency, dw to a Wiener increment (s.t. $dw^2 = dt$) and we have introduced the superoperator $\mathcal{H}[A]\varrho = A\varrho + \varrho A^\dagger - \mathrm{Tr}[\varrho(A + A^\dagger)]\varrho$. This conditional dynamics can be obtained, for instance, by considering the setup depicted in Fig. 1: a laser is collectively coupled to the total spin of the atoms (possibly inside a cavity) and the outcoming light is continuously measured after the interaction [51,61,62,90,105] (more details on these physical implementations are given in the Supplemental Material [73]).

When one considers these estimation strategies based on continuous measurements, with a dynamics obeying a SME such as Eq. (4), the parameter can be inferred from two sources of information: the continuous photocurrent dy_t and a final strong measurement on the conditional state ϱ_c. In this case the QFI $\mathcal{Q}$ in Eq. (2) is replaced by the so-called effective QFI [45],

$$\tilde{\mathcal{Q}}_{\mathsf{eff}} = \mathcal{F}_{y_t} + \sum_{\mathsf{traj}} p_{\mathsf{traj}}\mathcal{Q}[\varrho_c^{(\mathsf{traj})}], \quad (6)$$

that is the classical Fisher information (FI) $\mathcal{F}_{y_t}$ that quantifies the information obtainable from the continuous photocurrent dy_t, plus the average of the QFI of the conditional states $\varrho_c^{(\mathsf{traj})}$ corresponding to the different trajectories (more details in the Supplemental Material [73]). Furthermore, one can also consider the situation where the parameter is inferred from the continuous photocurrent dy_t only; in this scenario the appropriate bound is obtained by replacing $\mathcal{Q}$ with $\mathcal{F}_{y_t}$.

In the limit of a large number of atoms $N \gg 1$ and with no noise ($\kappa = 0$), it has been already demonstrated that, thanks to this measurement strategy, one can estimate the frequency ω with a Heisenberg-like scaling, despite the initial state being uncorrelated. The collective monitoring dynamically generates spin squeezing in the conditional states (and thus entanglement between the atoms), allowing one to observe an N^2 scaling both for the effective QFI and the classical FI [45].

Furthermore, in this work we consider a much more practical strategy than the one described in Refs. [71,72]. There, we have shown that the advantage of an initial entangled state can be recovered by monitoring the N independent environments responsible for the dephasing (typically inaccessible, in practice). Here, not only we consider a classical (separable) initial state, but we perform continuous monitoring on an ancillary quantum system over which we can assume to have full control; this may correspond, for instance, to an optical field, as depicted in Fig. 1.

Results.—The SME (4) is invariant under permutation of the different atoms. This symmetry can be exploited to dramatically reduce the dimension of the density operator ϱ_c as described in Refs. [93,106]. By exploiting some dedicated functions of QuTiP [94,107] introduced in Ref. [93], we have developed a code in Julia (available in Ref. [92]) that has allowed us to simulate quantum trajectories solving the SME (4) and to calculate the figures of merit introduced above up to $N = 150$ atoms (see Supplemental Material [73] for details on the numerics).

Before moving to the noisy case, we mention that we have been able to verify that for $\kappa = 0$ the estimation precision follows a Heisenberg scaling, not only in the limit $N \gg 1$, but also for nonasymptotic values of N: our numerics show that both the classical FI $\mathcal{F}_{y_t}$ and the average QFI $\bar{\mathcal{Q}}_c = \sum_{\mathrm{traj}} p_{\mathrm{traj}} \mathcal{Q}[\varrho_c^{(\mathrm{traj})}]$ (and thus their sum $\tilde{\mathcal{Q}}_{\mathrm{eff}}$) are quadratic in N (see Supplemental Material [73]).

We now focus on the effect of independent dephasing on this measurement strategy. In the upper panels of Fig. 2 we plot different figures of merit characterizing our strategy for $N = 50$ and $N = 100$, comparing them with the results obtained with CSS without monitoring. We observe that the effective QFI $\tilde{\mathcal{Q}}_{\mathrm{eff}}/t$ is larger than the CSS QFI $\mathcal{Q}_{\mathrm{CSS}}/t$ at all times. Remarkably, we observe that for $N = 100$ also the maximum of the monitoring FI $\max_t[\mathcal{F}_{y_t}/t]$ surpasses the maximum for the standard strategy $\max_t[\mathcal{Q}_{\mathrm{CSS}}/t]$. In general, this behavior is confirmed for different values of κ. This clearly shows that, by increasing N, the information obtained from the photocurrent dy_t is enough to achieve a higher precision than via coherent spin states without monitoring.

We also find that $\bar{\mathcal{Q}}_c/t$ is larger than $\mathcal{Q}_{\mathrm{CSS}}/t$ at certain times. This result can be explained by studying the spin-squeezing witness [48,51,108] $\zeta_y[\varrho] = [(\langle J_z\rangle^2 + \langle J_x\rangle^2)/N\Delta J_y^2]$, where $\langle A\rangle = \mathrm{Tr}[\varrho A]$ and $\Delta J_y^2 = \langle J_y^2\rangle - \langle J_y\rangle^2$. If $\zeta_y[\varrho] > 1$, the state ϱ is spin squeezed along the y direction. In the bottom panels of Fig. 2 we plot the average spin squeezing $\bar{\zeta}_y = \sum_{\mathrm{traj}} p_{\mathrm{traj}} \zeta_y[\varrho_c^{(\mathrm{traj})}]$ and indeed we observe the maximum violation approximately at the same time t for which $\bar{\mathcal{Q}}_c/t > \mathcal{Q}_{\mathrm{CSS}}/t$ (for more details about the distribution of trajectory dependent quantities, as $\zeta_y[\varrho_c^{(\mathrm{traj})}]$ and $\mathcal{Q}[\varrho_c^{(\mathrm{traj})}]$, see Supplemental Material [73]). The generation of spin-squeezed conditional states leads us to investigate the effectiveness of a strong measurement of the operator J_y, optimal in the noiseless case [45]. We evaluate the corresponding classical Fisher information, and we average it over the different trajectories, yielding $\bar{\mathcal{F}}_c[J_y]$. As shown in Fig. 2, $\bar{\mathcal{F}}_c[J_y]$ is approximately equal to $\bar{\mathcal{Q}}_c$ for evolution times near to the maximum of both $\bar{\mathcal{Q}}_c$ and $\tilde{\mathcal{Q}}_{\mathrm{eff}}$. In general, our numerics show that a strong measurement of J_y is nearly optimal in the parameter regime relevant for our protocol.

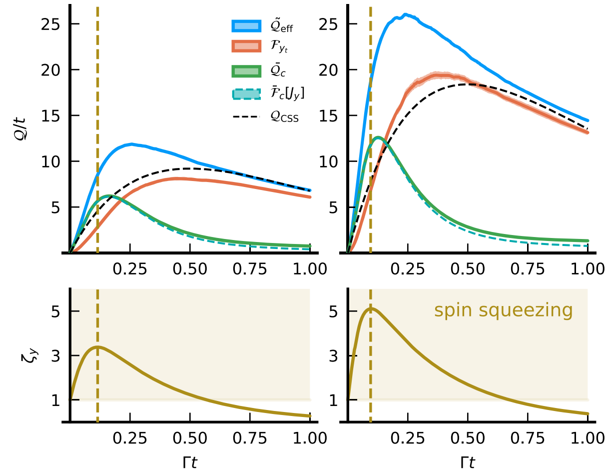

FIG. 2. Top: Information rate $\mathcal{Q}/t$ for noisy frequency estimation as a function of time in terms of different figures of merit. Blue line: effective QFI $\tilde{\mathcal{Q}}_{\mathrm{eff}}/t$; orange line: continuous monitoring classical FI $\mathcal{F}_{y_t}/t$; green line: conditional states average QFI $\bar{\mathcal{Q}}_c$; jade green dashed line: conditional states average FI for a J_y measurement $\bar{\mathcal{F}}_c[J_y]$; black dashed line: QFI for a CSS $\mathcal{Q}_{\mathrm{CSS}}/t$. Bottom: average spin squeezing $\bar{\zeta}_y$ as a function of time Γt. Left panels: $N = 50$; right panels: $N = 100$. The dashed vertical gold line corresponds to the monitoring time where the average spin squeezing violation is maximum. Parameters: $\kappa/\Gamma = 1$, $\omega/\Gamma = 10^{-2}$, $\eta = 1$, number of trajectories: $n_{\mathrm{traj}} = 15000$. The shaded areas represent the 95% confidence interval (see Supplemental Material [73]).

Importantly, the behavior of the spin-squeezing witness $\bar{\zeta}_y$ helps us also to better understand the optimal monitoring times for the different figures of merit plotted in Fig. 2. The following relationship holds:

$$t_{\mathrm{opt}}[\bar{\mathcal{Q}}_c] < t_{\mathrm{opt}}[\tilde{\mathcal{Q}}_{\mathrm{eff}}] < t_{\mathrm{opt}}[\mathcal{F}_{y_t}]. \tag{7}$$

In order to maximize the average QFI $\bar{\mathcal{Q}}_c/t$, as we discussed above, one therefore needs to stop the monitoring at a time $t_{\mathrm{opt}}[\bar{\mathcal{Q}}_c]$ corresponding approximately to the maximum spin squeezing. On the other hand, since $\mathcal{F}_{y_t}$ quantifies the information contained in the photocurrent y_t

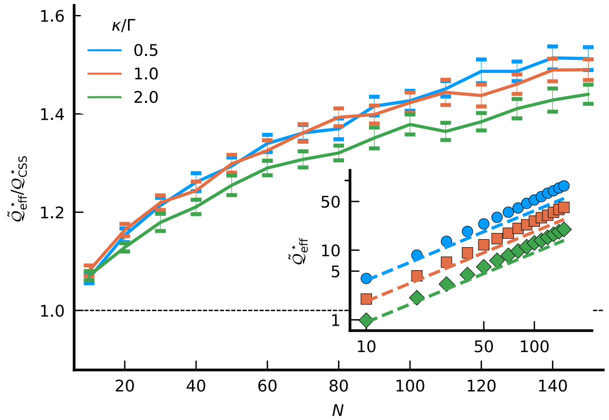

FIG. 3. Ratio between the optimized effective QFI $\tilde{\mathcal{Q}}^{\star}_{\rm eff}$ and the optimized $\mathcal{Q}^{\star}_{\rm CSS}$ (dashed lines) as a function of N for different values of the dephasing rate κ, with $\omega/\Gamma = 10^{-2}$ and $n_{\rm traj} = 10000$ trajectories. See the Supplemental Material [73] for details on the statistical error. In the inset, log-log plot of $\tilde{\mathcal{Q}}^{\star}_{\rm eff}$ (markers) and $\mathcal{Q}_{\rm CSS}$ (dashed lines) as a function of N for the same values of κ.

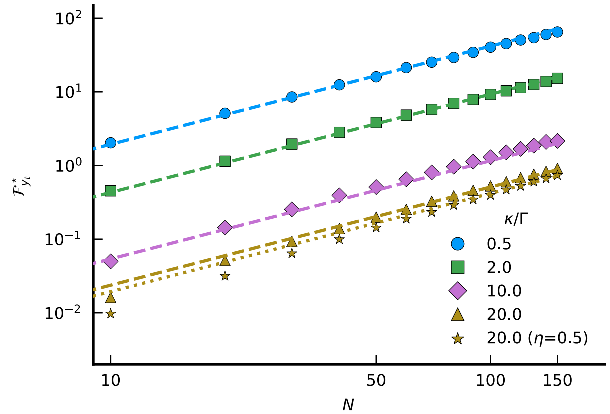

FIG. 4. Continuous monitoring FI $\max_t[\mathcal{F}_{y_t}/t]$ (markers) as a function of N for different values of the dephasing rate, with $\omega/\Gamma = 10^{-2}$ and number of trajectories $n_{\rm traj} = 10000$. Dashed lines showing superlinear functions scaling as $N^{4/3}$ have been plotted as a guide to the eye.

accumulated during the whole monitoring time, one can fully exploit the generated spin squeezing and the encoding of the parameter by waiting longer, i.e., $t_{\rm opt}[\mathcal{F}_{y_t}] > t_{\rm opt}[\bar{\mathcal{Q}}_c]$. Consequently, since the effective QFI $\tilde{\mathcal{Q}}_{\rm eff}$ is the sum of $\mathcal{F}_{y_t}$ and $\bar{\mathcal{Q}}_c$, the corresponding optimal time has to satisfy the relation in Eq. (7).

Figure 3 shows the ratio between the optimized effective QFI $\tilde{\mathcal{Q}}^{\star}_{\rm eff} \equiv \max_t[\tilde{\mathcal{Q}}_{\rm eff}/t]$ and the CSS bound $\mathcal{Q}^{*}_{\rm CSS}$ as a function of N and for different values of the dephasing rate κ. It is clear from the plot and from the inset, where the two quantities are plotted in logarithmic scale, that not only the CSS bound is always surpassed, but also the effective QFI shows a superlinear behavior. An important role in this result is played by the photocurrent FI $\mathcal{F}_{y_t}$, which corresponds to the most practical strategy of estimating ω without any strong final measurement. As it is apparent from Fig. 4, the behavior of $\mathcal{F}^{\star}_{y_t} \equiv \max_t[\mathcal{F}_{y_t}/t]$ is very peculiar: a κ-independent superclassical scaling $\mathcal{F}^{\star}_{y_t} \sim N^{4/3}$ seems to hold for all the considered values of the dephasing strength κ (notice that by increasing κ the scaling $N^{4/3}$ is obtained and then maintained for large enough N). It is also important to mention that a reduced measurement efficiency (e.g., $\eta = 0.5$ in one of the curves in Fig. 4) yields the same qualitative results as having a larger dephasing: (i) the CSS QFI is surpassed as long as N is large enough; (ii) despite nonunit efficiency, the κ-independent scaling $N^{4/3}$ is still observed for $\mathcal{F}_{y_t}$, but for larger N and with a reduced proportionality constant (more plots for $\eta < 1$ are found in the Supplemental Material [73]).

Finally, we consider the performance of our strategy in the presence of collective Markovian dephasing, that is, a dynamics described by a master equation as in Eq. (1), but with the last term replaced by $\kappa_{\rm coll}\mathcal{D}[J_z]\varrho$. Also in this case our scheme based on continuous QND monitoring performs better than a standard strategy with CSS states and no monitoring. However, we observe that spin squeezing is hardly generated and that no enhancement in the estimation precision due to quantum correlations can be observed (see more details in the Supplemental Material [73]). It is crucial to remark that collective dephasing is best tackled with specific estimation protocols, exploiting decoherence-free subspaces, that are able to restore Heisenberg scaling [79]. We therefore leave to future investigations the possibility of combining these strategies with our approach, to jointly counteract both independent and collective dephasing.

Discussion.—We showed that continuous QND monitoring leads to an enhancement in the estimation precision, even in the presence of Markovian dephasing, known to be the most detrimental noise for quantum metrology.

One last remark regarding the precision our protocol can ultimately achieve is in order. A fundamental bound that covers strategies with ancillary systems and full and fast control [100,101] shows that only an improvement of a factor e on $\mathcal{Q}^{\star}_{\rm CSS}$ in Eq. (3) can be obtained, i.e., $\mathcal{B}_{\rm ent} = N/(2\kappa)$. This bound is attained asymptotically for $N \gg 1$ by preparing a spin squeezed initial state, without ancillas and control operations [46]. At present it is not clear if the effective QFI for our scheme should also obey this bound. Continuous monitoring can be described as qubits interacting with the system and being sequentially measured [109–111]. However, it is unclear if the assumptions beyond the derivation in Refs. [100,101] are satisfied in the limit of infinitesimal time steps with simultaneous encoding, noise, and interaction with the ancillas. Despite the high optimization level of our code, we could not investigate regimes where our strategy would be able to reach values near to $\mathcal{B}_{\rm ent}$. We thus leave as an open question if our protocol, thanks in particular to the observed scaling of the classical FI $\mathcal{F}^*_{y_t}$, may be able to attain (or possibly surpass) this bound in experimentally relevant regimes (state of the art experiments with atomic clouds involve 10^5–10^{11} atoms [58–60]).

Finally, we highlight again one of the main features of our protocol: the monitoring-induced dynamics generates the resourceful state simultaneously with the frequency encoding. In fact, in the standard analysis of quantum estimation strategies the state preparation time is typically neglected. A fair comparison between "classical" and "quantum enhanced" strategies accounting also the preparation time as a resource is discussed, for the noiseless scenario, in Refs. [112–114]. In Refs. [113,114], in particular, the generation of spin squeezing via one -axis and two-axis twisting is considered and it is shown that the best strategy is to allow the encoding and the spin-squeezing Hamiltonians to act simultaneously. Remarkably enough, this enhancement is comparable to the one we observe in our protocol, with no need of time-dependent control Hamiltonians.

The role of preparation time in noisy metrology with Markovian independent dephasing has been discussed in Ref. [112]. There, however, only initial GHZ states have been considered and, unsurprisingly, they offer no improvement over CSS states; the same is true when the preparation time is not taken into account [65]. Optimal entangled states for standard frequency estimation in the presence of dephasing have been numerically obtained in Ref. [115]. We observe that, remarkably, our protocol can achieve an enhancement of the same order of magnitude [cf. Fig. 3 and Fig. 3(b) of Ref. [115]]. We therefore expect that, if the preparation time is counted as a resource, our protocol should be able to outperform the one involving the preparation of those optimal states.

Concluding, our results pave the way to further theoretical and experimental investigations into noisy quantum metrology via QND continuous monitoring, as a practical and relevant tool to obtain a quantum enhancement in spite of decoherence.

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[1] V. Giovannetti, S. Lloyd, and L. Maccone, Advances in quantum metrology, Nat. Photonics **5**, 222 (2011).
[2] D. Braun, G. Adesso, F. Benatti, R. Floreanini, U. Marzolino, M. W. Mitchell, and S. Pirandola, Quantum-enhanced measurements without entanglement, Rev. Mod. Phys. **90**, 035006 (2018).
[3] M. A. Taylor and W. P. Bowen, Quantum metrology and its application in biology, Phys. Rep. **615**, 1 (2016).
[4] C. M. Caves, Quantum-mechanical noise in an interferometer, Phys. Rev. D **23**, 1693 (1981).
[5] R. Demkowicz-Dobrzański, M. Jarzyna, and J. Kołodyński, Quantum limits in optical interferometry, Prog. Opt. **60**, 345 (2015).
[6] F. Acernese *et al.*, Increasing the Astrophysical Reach of the Advanced Virgo Detector via the Application of Squeezed Vacuum States of Light, Phys. Rev. Lett. **123**, 231108 (2019).
[7] M. Tse *et al.*, Quantum-Enhanced Advanced LIGO Detectors in the Era of Gravitational-Wave Astronomy, Phys. Rev. Lett. **123**, 231107 (2019).
[8] C. L. Degen, F. Reinhard, and P. Cappellaro, Quantum sensing, Rev. Mod. Phys. **89**, 035002 (2017).
[9] L. Pezzè, A. Smerzi, M. K. Oberthaler, R. Schmied, and P. Treutlein, Quantum metrology with nonclassical states of atomic ensembles, Rev. Mod. Phys. **90**, 035005 (2018).
[10] K. Bongs, M. Holynski, J. Vovrosh, P. Bouyer, G. Condon, E. Rasel, C. Schubert, W. P. Schleich, and A. Roura, Taking atom interferometric quantum sensors from the laboratory to real-world applications, Nat. Rev. Phys. **1**, 731 (2019).
[11] D. Budker and M. Romalis, Optical magnetometry, Nat. Phys. **3**, 227 (2007).
[12] M. Koschorreck, M. Napolitano, B. Dubost, and M. W. Mitchell, Sub-Projection-Noise Sensitivity in Broadband Atomic Magnetometry, Phys. Rev. Lett. **104**, 093602 (2010).
[13] W. Wasilewski, K. Jensen, H. Krauter, J. J. Renema, M. V. Balabas, and E. S. Polzik, Quantum Noise Limited and Entanglement-Assisted Magnetometry, Phys. Rev. Lett. **104**, 133601 (2010).
[14] R. J. Sewell, M. Koschorreck, M. Napolitano, B. Dubost, N. Behbood, and M. W. Mitchell, Magnetic Sensitivity Beyond the Projection Noise Limit by Spin Squeezing, Phys. Rev. Lett. **109**, 253605 (2012).
[15] F. Troiani and M. G. A. Paris, Universal Quantum Magnetometry with Spin States at Equilibrium, Phys. Rev. Lett. **120**, 260503 (2018).
[16] A. D. Ludlow, M. M. Boyd, J. Ye, E. Peik, and P. O. Schmidt, Optical atomic clocks, Rev. Mod. Phys. **87**, 637 (2015).
[17] A. Louchet-Chauvet, J. Appel, J. J. Renema, D. Oblak, N. Kjaergaard, and E. S. Polzik, Entanglement-assisted atomic clock beyond the projection noise limit, New J. Phys. **12**, 065032 (2010).
[18] E. M. Kessler, P. Kómár, M. Bishof, L. Jiang, A. S. Sørensen, J. Ye, and M. D. Lukin, Heisenberg-Limited Atom Clocks Based on Entangled Qubits, Phys. Rev. Lett. **112**, 190403 (2014).

[19] H. M. Wiseman and G. J. Milburn, *Quantum Measurement and Control* (Cambridge University Press, New York, 2010).

[20] K. Jacobs, *Quantum Measurement Theory and its Applications* (Cambridge University Press, Cambridge, England, 2014).

[21] K. W. Murch, S. J. Weber, C. Macklin, and I. Siddiqi, Observing single quantum trajectories of a superconducting quantum bit, Nature (London) **502**, 211 (2013).

[22] Q. Ficheux, S. Jezouin, Z. Leghtas, and B. Huard, Dynamics of a qubit while simultaneously monitoring its relaxation and dephasing, Nat. Commun. **9**, 1926 (2018).

[23] Z. K. Minev, S. O. Mundhada, S. Shankar, P. Reinhold, R. Gutiérrez-Jáuregui, R. J. Schoelkopf, M. Mirrahimi, H. J. Carmichael, and M. H. Devoret, To catch and reverse a quantum jump mid-flight, Nature (London) **570**, 200 (2019).

[24] W. Wieczorek, S. G. Hofer, J. Hoelscher-Obermaier, R. Riedinger, K. Hammerer, and M. Aspelmeyer, Optimal State Estimation for Cavity Optomechanical Systems, Phys. Rev. Lett. **114**, 223601 (2015).

[25] M. Rossi, D. Mason, J. Chen, and A. Schliesser, Observing and Verifying the Quantum Trajectory of a Mechanical Resonator, Phys. Rev. Lett. **123**, 163601 (2019).

[26] R. A. Thomas, M. Parniak, C. Østfeldt, C. B. Møller, C. Bærentsen, Y. Tsaturyan, A. Schliesser, J. Appel, E. Zeuthen, and E. S. Polzik, Entanglement between distant macroscopic mechanical and spin systems, arXiv:2003.11310.

[27] H. Mabuchi, Dynamical identification of open quantum systems, Quantum Semiclass. Opt. **8**, 1103 (1996).

[28] J. Gambetta and H. M. Wiseman, State and dynamical parameter estimation for open quantum systems, Phys. Rev. A **64**, 042105 (2001).

[29] J. M. Geremia, J. K. Stockton, A. C. Doherty, and H. Mabuchi, Quantum Kalman Filtering and the Heisenberg Limit in Atomic Magnetometry, Phys. Rev. Lett. **91**, 250801 (2003).

[30] M. Tsang, Optimal waveform estimation for classical and quantum systems via time-symmetric smoothing. II. Applications to atomic magnetometry and Hardy's paradox, Phys. Rev. A **81**, 013824 (2010).

[31] R. L. Cook, C. A. Riofrío, and I. H. Deutsch, Single-shot quantum state estimation via a continuous measurement in the strong backaction regime, Phys. Rev. A **90**, 032113 (2014).

[32] P. Six, P. Campagne-Ibarcq, L. Bretheau, B. Huard, and P. Rouchon, Parameter estimation from measurements along quantum trajectories, in *2015 54th IEEE Conf. Decis. Control*, Cdc (IEEE, New York, 2015), p. 7742.

[33] A. H. Kiilerich and K. Mølmer, Bayesian parameter estimation by continuous homodyne detection, Phys. Rev. A **94**, 032103 (2016).

[34] L. Cortez, A. Chantasri, L. P. García-Pintos, J. Dressel, and A. N. Jordan, Rapid estimation of drifting parameters in continuously measured quantum systems, Phys. Rev. A **95**, 012314 (2017).

[35] J. F. Ralph, S. Maskell, and K. Jacobs, Multiparameter estimation along quantum trajectories with sequential Monte Carlo methods, Phys. Rev. A **96**, 052306 (2017).

[36] J. Atalaya, S. Hacohen-Gourgy, L. S. Martin, I. Siddiqi, and A. N. Korotkov, Correlators in simultaneous measurement of non-commuting qubit observables, npj Quantum Inf. **4**, 41 (2018).

[37] A. Shankar, G. P. Greve, B. Wu, J. K. Thompson, and M. Holland, Continuous Real-Time Tracking of a Quantum Phase below the Standard Quantum Limit, Phys. Rev. Lett. **122**, 233602 (2019).

[38] M. Guţă, B. Janssens, and J. Kahn, Optimal estimation of qubit states with continuous time measurements, Commun. Math. Phys. **277**, 127 (2007).

[39] M. Tsang, H. M. Wiseman, and C. M. Caves, Fundamental Quantum Limit to Waveform Estimation, Phys. Rev. Lett. **106**, 090401 (2011).

[40] M. Tsang, Quantum metrology with open dynamical systems, New J. Phys. **15**, 073005 (2013).

[41] S. Gammelmark and K. Mølmer, Bayesian parameter inference from continuously monitored quantum systems, Phys. Rev. A **87**, 032115 (2013).

[42] S. Gammelmark and K. Mølmer, Fisher Information and the Quantum Cramér-Rao Sensitivity Limit of Continuous Measurements, Phys. Rev. Lett. **112**, 170401 (2014).

[43] M. Guţă and J. Kiukas, Information geometry and local asymptotic normality for multi-parameter estimation of quantum Markov dynamics, J. Math. Phys. (N.Y.) **58**, 052201 (2017).

[44] M. G. Genoni, Cramér-Rao bound for time-continuous measurements in linear Gaussian quantum systems, Phys. Rev. A **95**, 012116 (2017).

[45] F. Albarelli, M. A. C. Rossi, M. G. A. Paris, and M. G. Genoni, Ultimate limits for quantum magnetometry via time-continuous measurements, New J. Phys. **19**, 123011 (2017).

[46] D. Ulam-Orgikh and M. Kitagawa, Spin squeezing and decoherence limit in Ramsey spectroscopy, Phys. Rev. A **64**, 052106 (2001).

[47] M. Kitagawa and M. Ueda, Squeezed spin states, Phys. Rev. A **47**, 5138 (1993).

[48] J. Ma, X. Wang, C.-P. Sun, and F. Nori, Quantum spin squeezing, Phys. Rep. **509**, 89 (2011).

[49] Y. Takahashi, K. Honda, N. Tanaka, K. Toyoda, K. Ishikawa, and T. Yabuzaki, Quantum nondemolition measurement of spin via the paramagnetic Faraday rotation, Phys. Rev. A **60**, 4974 (1999).

[50] A. Kuzmich, L. Mandel, and N. P. Bigelow, Generation of Spin Squeezing via Continuous Quantum Nondemolition Measurement, Phys. Rev. Lett. **85**, 1594 (2000).

[51] L. K. Thomsen, S. Mancini, and H. M. Wiseman, Spin squeezing via quantum feedback, Phys. Rev. A **65**, 061801 (R) (2002).

[52] K. Hammerer, A. S. Sørensen, and E. S. Polzik, Quantum interface between light and atomic ensembles, Rev. Mod. Phys. **82**, 1041 (2010).

[53] J. Appel, P. J. Windpassinger, D. Oblak, U. B. Hoff, N. Kjaergaard, and E. S. Polzik, Mesoscopic atomic entanglement for precision measurements beyond the standard quantum limit, Proc. Natl. Acad. Sci. U.S.A. **106**, 10960 (2009).

[54] M. H. Schleier-Smith, I. D. Leroux, and V. Vuletić, States of an Ensemble of Two-Level Atoms with Reduced

Quantum Uncertainty, Phys. Rev. Lett. **104**, 073604 (2010).

[55] V. Shah, G. Vasilakis, and M. V. Romalis, High Bandwidth Atomic Magnetometery with Continuous Quantum Nondemolition Measurements, Phys. Rev. Lett. **104**, 013601 (2010).

[56] J. G. Bohnet, K. C. Cox, M. A. Norcia, J. M. Weiner, Z. Chen, and J. K. Thompson, Reduced spin measurement back-action for a phase sensitivity ten times beyond the standard quantum limit, Nat. Photonics **8**, 731 (2014).

[57] N. Behbood, F. Martin Ciurana, G. Colangelo, M. Napolitano, G. Tóth, R. J. Sewell, and M. W. Mitchell, Generation of Macroscopic Singlet States in a Cold Atomic Ensemble, Phys. Rev. Lett. **113**, 093601 (2014).

[58] G. Vasilakis, H. Shen, K. Jensen, M. Balabas, D. Salart, B. Chen, and E. S. Polzik, Generation of a squeezed state of an oscillator by stroboscopic back-action-evading measurement, Nat. Phys. **11**, 389 (2015).

[59] O. Hosten, N. J. Engelsen, R. Krishnakumar, and M. A. Kasevich, Measurement noise 100 times lower than the quantum-projection limit using entangled atoms, Nature (London) **529**, 505 (2016).

[60] H. Bao, J. Duan, S. Jin, X. Lu, P. Li, W. Qu, M. Wang, I. Novikova, E. E. Mikhailov, K.-f. Zhao, K. Mølmer, H. Shen, and Y. Xiao, Spin squeezing of 10^{11} atoms by prediction and retrodiction measurements, Nature (London) **581**, 159 (2020).

[61] K. Mølmer and L. B. Madsen, Estimation of a classical parameter with Gaussian probes: Magnetometry with collective atomic spins, Phys. Rev. A **70**, 052102 (2004).

[62] L. B. Madsen and K. Mølmer, Spin squeezing and precision probing with light and samples of atoms in the Gaussian description, Phys. Rev. A **70**, 052324 (2004).

[63] J. K. Stockton, J. M. Geremia, A. C. Doherty, and H. Mabuchi, Robust quantum parameter estimation: Coherent magnetometry with feedback, Phys. Rev. A **69**, 032109 (2004).

[64] B. A. Chase, B. Q. Baragiola, H. L. Partner, B. D. Black, and J. M. Geremia, Magnetometry via a double-pass continuous quantum measurement of atomic spin, Phys. Rev. A **79**, 062107 (2009).

[65] S. F. Huelga, C. Macchiavello, T. Pellizzari, A. K. Ekert, M. B. Plenio, and J. I. Cirac, Improvement of Frequency Standards with Quantum Entanglement, Phys. Rev. Lett. **79**, 3865 (1997).

[66] A. Fujiwara and H. Imai, A fibre bundle over manifolds of quantum channels and its application to quantum statistics, J. Phys. A **41**, 255304 (2008).

[67] B. M. Escher, R. L. de Matos Filho, and L. Davidovich, General framework for estimating the ultimate precision limit in noisy quantum-enhanced metrology, Nat. Phys. **7**, 406 (2011).

[68] R. Demkowicz-Dobrzański, J. Kołodyński, and M. Guţă, The elusive Heisenberg limit in quantum-enhanced metrology, Nat. Commun. **3**, 1063 (2012).

[69] T. Gefen, D. A. Herrera-Martí, and A. Retzker, Parameter estimation with efficient photodetectors, Phys. Rev. A **93**, 032133 (2016).

[70] M. B. Plenio and S. F. Huelga, Sensing in the presence of an observed environment, Phys. Rev. A **93**, 032123 (2016).

[71] F. Albarelli, M. A. C. Rossi, D. Tamascelli, and M. G. Genoni, Restoring Heisenberg scaling in noisy quantum metrology by monitoring the environment, Quantum **2**, 110 (2018).

[72] F. Albarelli, M. A. C. Rossi, and M. G. Genoni, Quantum frequency estimation with conditional states of continuously monitored independent dephasing channels, Int. J. Quantum. Inform. **17**, 1941013 (2019).

[73] See Supplemental Material http://link.aps.org/supplemental/10.1103/PhysRevLett.125.200505 for an overview of quantum estimation theory for continuous measurements, details on the physical implementation of the stochastic master equation, details on the numerical implementation and additional data for noiseless dynamics, nonunit efficiency monitoring, and for collective dephasing, which includes Refs. [74–97].

[74] C. W. Helstrom, *Quantum Detection and Estimation Theory* (Academic Press, New York, 1976).

[75] A. S. Holevo, *Probabilistic and Statistical Aspects of Quantum Theory*, 2nd ed. (Edizioni della Normale, Pisa, 2011).

[76] *Asymptotic Theory of Quantum Statistical Inference*, edited by M. Hayashi (World Scientific, Singapore, 2005).

[77] S. L. Braunstein and C. M. Caves, Statistical Distance and the Geometry of Quantum States, Phys. Rev. Lett. **72**, 3439 (1994).

[78] M. G. A. Paris, Quantum estimation for quantum technology, Int. J. Quantum. Inform. **07**, 125 (2009).

[79] U. Dorner, Quantum frequency estimation with trapped ions and atoms, New J. Phys. **14**, 043011 (2012).

[80] O. E. Barndorff-Nielsen and R. D. Gill, Fisher information in quantum statistics, J. Phys. A **33**, 4481 (2000).

[81] M. A. C. Rossi, F. Albarelli, and M. G. A. Paris, Enhanced estimation of loss in the presence of Kerr nonlinearity, Phys. Rev. A **93**, 053805 (2016).

[82] D. Tamascelli, C. Benedetti, H.-P. Breuer, and M. G. A. Paris, Quantum probing beyond pure dephasing, New J. Phys. **22**, 083027 (2020).

[83] C. Catana and M. Guţă, Heisenberg versus standard scaling in quantum metrology with markov generated states and monitored environment, Phys. Rev. A **90**, 012330 (2014).

[84] K. Macieszczak, M. Guţă, I. Lesanovsky, and J. P. Garrahan, Dynamical phase transitions as a resource for quantum enhanced metrology, Phys. Rev. A **93**, 022103 (2016).

[85] P. Rouchon and J. F. Ralph, Efficient quantum filtering for quantum feedback control, Phys. Rev. A **91**, 012118 (2015).

[86] K. Jacobs and D. A. Steck, A straightforward introduction to continuous quantum measurement, Contemp. Phys. **47**, 279 (2006).

[87] L. K. Thomsen, S. Mancini, and H. M. Wiseman, Continuous quantum nondemolition feedback and unconditional atomic spin squeezing, J. Phys. B **35**, 4937 (2002).

[88] I. H. Deutsch and P. S. Jessen, Quantum control and measurement of atomic spins in polarization spectroscopy, Opt. Commun. **283**, 681 (2010).

[89] J. Gough and R. van Handel, Singular perturbation of quantum stochastic differential equations with coupling through an oscillator mode, J. Stat. Phys. **127**, 575 (2007).

[90] L. Bouten, J. Stockton, G. Sarma, and H. Mabuchi, Scattering of polarized laser light by an atomic gas in free space: A quantum stochastic differential equation approach, Phys. Rev. A **75**, 052111 (2007).

[91] J. Bezanson, A. Edelman, S. Karpinski, and V. B. Shah, Julia: A fresh approach to numerical computing, SIAM Rev. **59**, 65 (2017).

[92] M. A. C. Rossi, F. Albarelli, D. Tamascelli, and M. G. Genoni, QContinuousMeasurement.jl, https://github.com/matteoacrossi/QContinuousMeasurement.jl (2020).

[93] N. Shammah, S. Ahmed, N. Lambert, S. De Liberato, and F. Nori, Open quantum systems with local and collective incoherent processes: Efficient numerical simulations using permutational invariance, Phys. Rev. A **98**, 063815 (2018).

[94] J. Johansson, P. Nation, and F. Nori, QuTiP: An open-source PYTHON framework for the dynamics of open quantum systems, Comput. Phys. Commun. **183**, 1760 (2012).

[95] M. A. C. Rossi, F. Albarelli, D. Tamascelli, and M. G. Genoni, Noisy quantum metrology enhanced by continuous nondemolition measurement [Data set], https://doi.org/10.5281/zenodo.4109173 (2020), Zenodo.

[96] D. de Falco and D. Tamascelli, Noise-assisted quantum transport and computation, J. Phys. A **46**, 225301 (2013).

[97] G. S. Agarwal, Relation between atomic coherent-state representation, state multipoles, and generalized phase-space distributions, Phys. Rev. A **24**, 2889 (1981).

[98] R. Chaves, J. B. Brask, M. Markiewicz, J. Kołodyński, and A. Acín, Noisy Metrology beyond the Standard Quantum Limit, Phys. Rev. Lett. **111**, 120401 (2013).

[99] J. B. Brask, R. Chaves, and J. Kołodyński, Improved Quantum Magnetometry beyond the Standard Quantum Limit, Phys. Rev. X **5**, 031010 (2015).

[100] P. Sekatski, M. Skotiniotis, J. Kołodyński, and W. Dür, Quantum metrology with full and fast quantum control, Quantum **1**, 27 (2017).

[101] R. Demkowicz-Dobrzański, J. Czajkowski, and P. Sekatski, Adaptive Quantum Metrology under General Markovian Noise, Phys. Rev. X **7**, 041009 (2017).

[102] S. Zhou, M. Zhang, J. Preskill, and L. Jiang, Achieving the Heisenberg limit in quantum metrology using quantum error correction, Nat. Commun. **9**, 78 (2018).

[103] A. W. Chin, S. F. Huelga, and M. B. Plenio, Quantum Metrology in Non-Markovian Environments, Phys. Rev. Lett. **109**, 233601 (2012).

[104] A. Smirne, J. Kołodyński, S. F. Huelga, and R. Demkowicz-Dobrzański, Ultimate Precision Limits for Noisy Frequency Estimation, Phys. Rev. Lett. **116**, 120801 (2016).

[105] A. E. B. Nielsen and K. Mølmer, Atomic spin squeezing in an optical cavity, Phys. Rev. A **77**, 063811 (2008).

[106] B. A. Chase and J. M. Geremia, Collective processes of an ensemble of spin 1/2 particles, Phys. Rev. A **78**, 052101 (2008).

[107] J. Johansson, P. Nation, and F. Nori, QuTiP 2: A PYTHON framework for the dynamics of open quantum systems, Comput. Phys. Commun. **184**, 1234 (2013).

[108] A. Sørensen, L. M. Duan, J. I. Cirac, and P. Zoller, Many-particle entanglement with Bose-Einstein condensates, Nature (London) **409**, 63 (2001).

[109] T. A. Brun, A simple model of quantum trajectories, Am. J. Phys. **70**, 719 (2002).

[110] F. Ciccarello, Collision models in quantum optics, Quantum Meas. Quantum Metrol. **4**, 53 (2017).

[111] J. A. Gross, C. M. Caves, G. J. Milburn, and J. Combes, Qubit models of weak continuous measurements: Markovian conditional and open-system dynamics, Quantum Sci. Technol. **3**, 024005 (2018).

[112] S. Dooley, W. J. Munro, and K. Nemoto, Quantum metrology including state preparation and readout times, Phys. Rev. A **94**, 052320 (2016).

[113] A. J. Hayes, S. Dooley, W. J. Munro, K. Nemoto, and J. Dunningham, Making the most of time in quantum metrology: Concurrent state preparation and sensing, Quantum Sci. Technol. **3**, 035007 (2018).

[114] S. A. Haine and J. J. Hope, Machine-Designed Sensor to Make Optimal Use of Entanglement-Generating Dynamics for Quantum Sensing, Phys. Rev. Lett. **124**, 060402 (2020).

[115] F. Fröwis, M. Skotiniotis, B. Kraus, and W. Dür, Optimal quantum states for frequency estimation, New J. Phys. **16**, 083010 (2014).